Approximating Cumulative Pebbling Cost is Unique Games Hard

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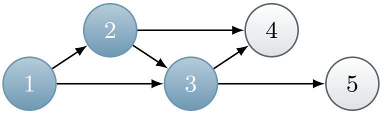
Contents

We are now at...

Graph Pebbling (Sequential/Parallel)

Consider a directed acyclic graph (DAG) $G = (V, E)$.



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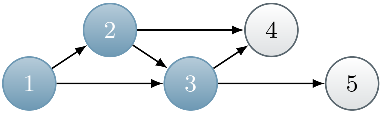

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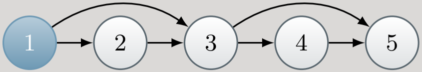

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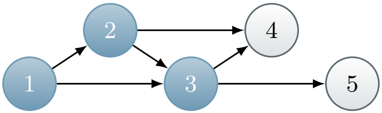

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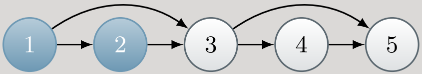

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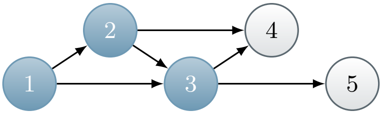

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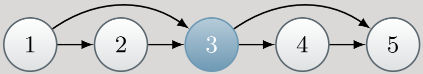

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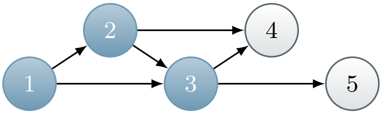

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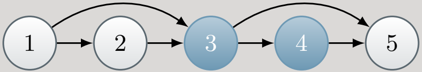

$P_4 = \{3, 4\}$ (data values L_3 and L_4 stored in memory)

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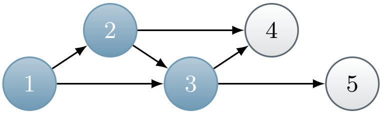

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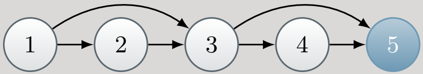

$P_5 = \{5\}$ (data value L_5 stored in memory)

Pebbling Complexity: The Cumulative Pebbling Cost $\mathsf{cc}(G)$

Let $\mathcal{P}_G^{\parallel}$ be the set of all valid ***parallel*** pebblings of G.

Definition

- The ***cumulative cost of a pebbling*** $P = (P_1, \cdots, P_t) \in \mathcal{P}_G^{\parallel}$ is

$$\mathsf{cc}(P) := |P_1| + \cdots + |P_t|.$$

- The ***cumulative pebbling cost of a graph*** G is defined by

$$\mathsf{cc}(G) = \min_{P \in \mathcal{P}_G^{\parallel}} \mathsf{cc}(P)$$

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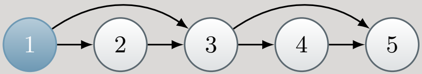

$$\mathsf{cc}(G) \le |P_1| + \cdots + |P_5| = 1 + 2 + 1 + 2 + 1 = 7.$$

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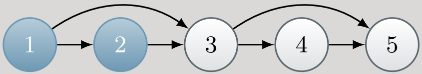

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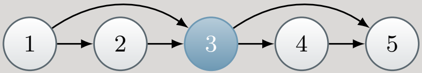

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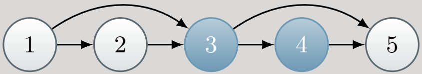

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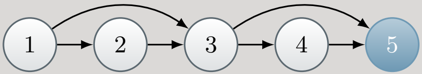

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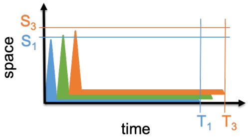
Applications of $\mathrm{cc}(G)$

- Password hashing - Memory Hard Function (MHF) f
- A brute-force attacker wants to compute f_G on many inputs (m)
    - Involves pebbling a DAG G m times
    - Want total cost as large as possible
- Consider the Space×Time (ST)-Complexity $\mathrm{ST}(G) := \min_{P \in \mathcal{P}_G^{\parallel}} \left(t_P \times \max_{i \le t_P} |P_i|\right)$.



$\mathrm{ST}_1 = S_1 \times T_1 \approx S_3 \times T_3 = \mathrm{ST}_3$

cost of computing *f* once — cost of computing *f* three times

Theorem [AS15] (informal)

For a secure side channel resistant memory hard function for password hashing, it suffices to find a DAG G with ***constant indegree*** and ***maximum*** $\mathrm{cc}(G)$.

- Cryptanalysis of MHF $\Leftrightarrow$ Find $\mathrm{cc}(G)$.

We are now at...

The Main Result

- Blocki and Zhou [BZ18] recently showed that computing $\mathrm{cc}(G)$ is NP-Hard. However, this does not rule out the existence of a $(1+\varepsilon)$-approximation algorithm for any constant $\varepsilon > 0$.
- Our main result is the hardness of any constant factor approximation to the cost of graph pebbling even for DAGs with constant indegree.

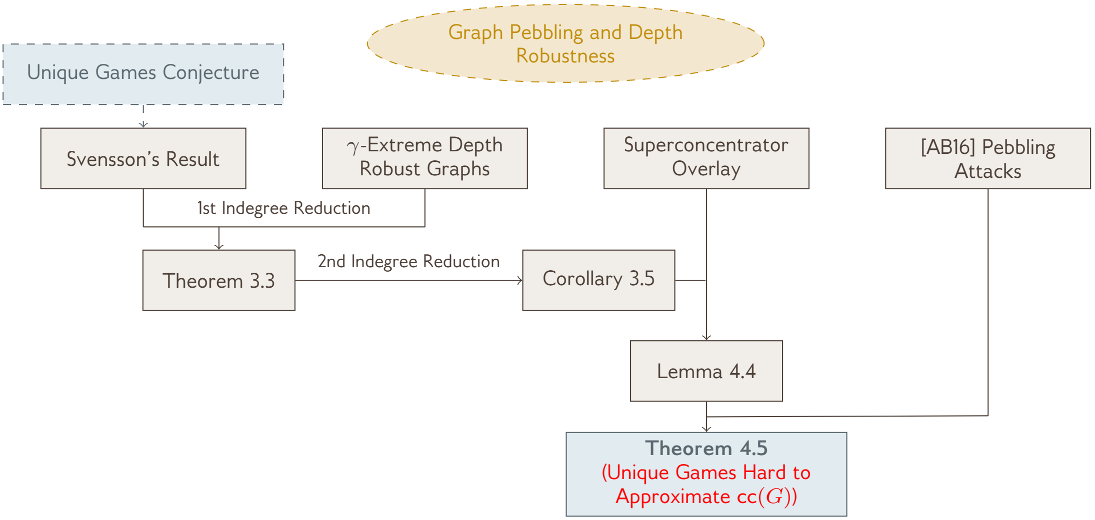
Theorem

Given a DAG G with constant indegree, it is Unique Games hard to approximate $\mathrm{cc}(G)$ within any constant factor.

Strategy?

- Svensson's result of Unique Games hardness to distinguish two cases for a DAG G
- Reduction to $\widetilde{G}$ with ***gap*** between the upper and lower bound of $\mathrm{cc}(\widetilde{G})$

Proof Overview



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Unique Games Conjecture

Definition (Unique Games)

An instance of Unique Games $\mathcal{U} = (G = (V, W, E), [R], \{\pi_{v,w}\}_{v,w})$ consists of a regular bipartite graph $G(V, W, E)$ and a set $[R]$ of labels. Each edge $(v, w) \in E$ has a constraint given by a permutation $\pi_{v,w} : [R] \to [R]$. The goal is to output a labeling $\rho : (V \cup W) \to [R]$ that maximizes the number of satisfied edges, where an edge is satisfied if $\rho(v) = \pi_{v,w}(\rho(w))$.

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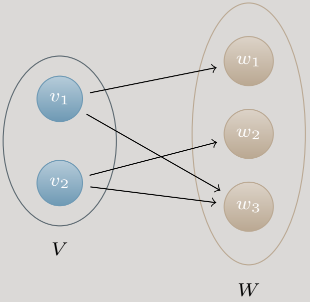

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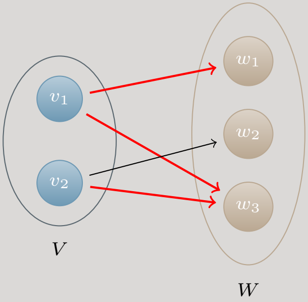

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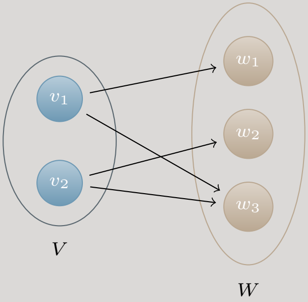

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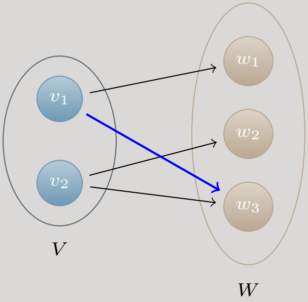

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The following conjecture from [Kho02] has been extensively used to prove several strong hardness of approximation algorithm.

Conjecture (Unique Games Conjecture) [Kho02]

For any constants $\alpha, \beta > 0$, there exists a sufficiently large integer R (as a function of α, β) such that for Unique Games instance with label set $[R]$, no polynomial time algorithm can distinguish whether:

1. (completeness) the maximum fraction of satisfied edges of any labeling is at least $1 - \alpha$, or
2. (soundness) the maximum fraction of satisfied edges of any labeling is less than β.

- Approximation algorithm for cc(G)?

We are now at...

Technical Ingredients 0: Depth Robustness ($\leftrightarrow$ Depth Reducibility)

First, we define depth(G) to be the length of the longest directed path in a DAG G.

Definition

- A DAG $G = (V, E)$ is (e, d)-***depth robust*** if

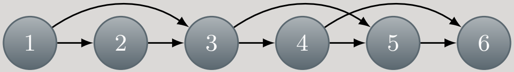
$$\forall S \subseteq V \text{ s.t. } |S| \leq e \quad \Rightarrow \quad \text{depth}(G - S) \geq d.$$

- We say that G is (e, d)-***reducible*** if G is not (e, d)-depth robust. That is,

$$\exists S \subseteq V \text{ s.t. } |S| \leq e \text{ and depth}(G - S) < d.$$

Example

The following graph is $(e = 2, d = 2)$-reducible:



Technical Ingredients 0: Depth Robustness ($\leftrightarrow$ Depth Reducibility)

First, we define depth(G) to be the length of the longest directed path in a DAG G.

Definition

- A DAG $G = (V, E)$ is (e, d)-***depth robust*** if

$$\forall S \subseteq V \text{ s.t. } |S| \leq e \quad \Rightarrow \quad \text{depth}(G - S) \geq d.$$

- We say that G is (e, d)-***reducible*** if G is not (e, d)-depth robust. That is,

$$\exists S \subseteq V \text{ s.t. } |S| \leq e \text{ and depth}(G - S) < d.$$

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A few facts about depth robustness:

- [AB16] For any (e, d)-reducible DAG G with N nodes,

$$\text{cc}(G) \leq \min_{g \geq d} \left(eN + gN \times \text{indeg}(G) + \frac{N^2 d}{g} \right).$$

- [ABP17] For any (e, d)-depth robust DAG G,

$$\text{cc}(G) \geq ed.$$

We are now at...

Technical Ingredients 1: Svensson's Result of Unique Game Hardness

Svensson [Sve12] proved the Unique Games hardness of a DAG G:

Theorem [Sve12]

For any constant $k, \varepsilon > 0$, it is Unique Games hard to distinguish between whether

1. G is (e_1, d_1)-reducible with $e_1 = N/k$ and $d_1 = k$, and
2. G is (e_2, d_2)-depth robust with $e_2 = N(1 - 1/k)$ and $d_2 = \Omega(N^{1-\varepsilon})$.

- To prove this, reduction from an instance of Unique Games $\mathcal{U} = (G = (V, W, E), [R], \{\pi_{v,w}\}_{v,w})$ to a DAG $G_{\mathcal{U}}$ on N nodes.
    - G is (e_1, d_1)-reducible if $\mathcal{U}$ is satisfiable, and
    - G is (e_2, d_2)-depth robust if $\mathcal{U}$ is unsatisfiable.
- As mentioned before, we have nice upper and lower bounds for $\mathrm{cc}(G)$ from [ABP17] and [AB16]:

Theorem

- [ABP17] For any (e, d)-depth robust DAG G, we have $\mathrm{cc}(G) \geq ed$.
- [AB16] For any (e, d)-reducible DAG G with N nodes, we have $\mathrm{cc}(G) \leq \min_{g \geq d} \left(eN + gN \times \mathrm{indeg}(G) + \frac{N^2 d}{g} \right)$.

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- Why can't we directly use this result to obtain our result (UG hardness of approximability of cc(G)?)

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Challenges of Applying Svensson's Construction

The result of Alwen et al. [ABP17] and [AB16] tells us that

- $\mathsf{cc}(G_{\mathcal{U}}) \geq e_2 d_2$, and
- $\mathsf{cc}(G_{\mathcal{U}}) \leq \min_{g \geq d_1} \left(e_1 N + gN \times \mathsf{indeg}(G_{\mathcal{U}}) + \frac{N^2 d_1}{g} \right)$

$\Rightarrow$ no gap between the upper/lower bounds since $\mathsf{indeg}(G_{\mathcal{U}}) = \mathcal{O}(N)$ implies

$$gN \times \mathsf{indeg}(G_{\mathcal{U}}) = gN^2 \gg e_2 d_2.$$

$\Rightarrow$ need to reduce the indegree (how? using γ-extreme depth-robust graphs.)

1. The graph $\hat{G}_{\mathcal{U}}$ contains two types of vertices:
   - bit-vertices partitioned into bit-layers $B = B_0 \cup \cdots \cup B_L$,
   - test-vertices partitioned into test-layers $T = T_0 \cup \cdots \cup T_{L-1}$, and
   - all of the edges in the graph are between bit-vertices and test-vertices.

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2. $\hat{G}_{\mathcal{U}}$ shows symmetry between the layers:
   - $B_\ell = \{b_1^\ell, \cdots, b_m^\ell\}$ and $T_\ell = \{t_1^\ell, \cdots, t_p^\ell\}$ (# of bit- and test-vertices in each layer is the same)
   - The edges between B_ℓ and T_ℓ (resp. T_ℓ and $B_{\ell+1}$) encode the edge constraints in the UG instance $\mathcal{U}$.
   - The directed edge (b_i^ℓ, t_j^ℓ) exists $\Leftrightarrow \forall \ell' \geq \ell$ the edge $(b_i^\ell, t_j^{\ell'})$ exists.
   - The directed edge $(t_j^\ell, b_i^{\ell+1})$ exists $\Leftrightarrow \forall \ell' > \ell$ the edge $(t_j^\ell, b_i^{\ell'})$ exists.

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3. The number of layers $L = N^{1-\varepsilon}$.

Svensson's Construction

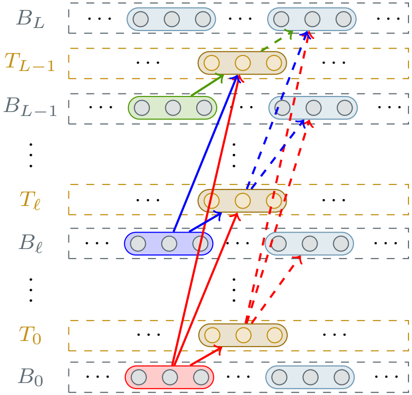
- $C_{x,S} = \{z \in [k]^R : z_j = x_j \ \forall j \notin S\}$, (the sub-cube whose coordinates not in S are fixed according to x)
- $C_{x,S,v,w} = \{z \in [k]^R : z_j = x_{\pi_{v,w}(j)} \ \forall \pi_{v,w}(j) \notin S\}$, (the image of the sub-cube $C_{x,S}$ under $\pi_{v,w}$)
- $C^{\oplus}_{x,S} = \{z \oplus 1 : z \in C_{x,S}\}$, (where $\oplus$ denotes coordinate-wise addition modulo k)
- $C^{\oplus}_{x,S,v,w} = \{z \oplus 1 : z \in C_{x,S,v,w}\}$.

Svensson's Construction for the Graph $\hat{G}_{\mathcal{U}}$

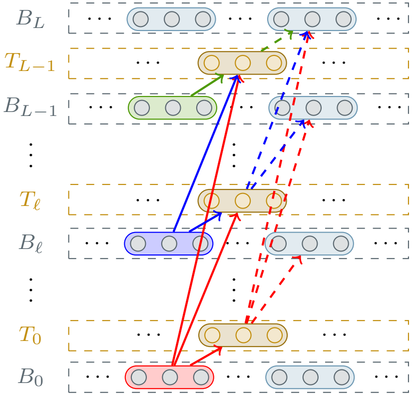
We fix k to be an integer and $\varepsilon, \delta > 0$ to be arbitratily small constants. For some L to be fixed,

- There are $L+1$ layers of bit-vertices. Each set of bit-vertices B_ℓ with $0 \le \ell \le L$ contains $b^{\ell}_{w,x}$ for each $w \in W$ and $x \in [k]^R$.
- There are L layers of test-vertices. Each set of test-vertices T_ℓ with $0 \le \ell \le L-1$ contains $t^{\ell}_{x,S,v,w_1,\cdots,w_{2t}}$ for each $x \in [k]^R, S = (s_1, \cdots, s_{\varepsilon R}) \in [R]^{\varepsilon R}, v \in V$ and every sequence of $(w_1, \cdots, w_{2t})$ (not necessarily distinct) neighbors of v.
- If $\ell \le \ell'$ and $z \in C_{x,S,v,w_j}$, then add an edge from $b^{\ell}_{w_j,z}$ to $t^{\ell'}_{x,S,v,w_1,\cdots,w_{2t}}$ for each $1 \le j \le 2t$.
- If $\ell > \ell'$ and $z \in C^{\oplus}_{x,S,v,w_j}$, then add an edge from $t^{\ell'}_{x,S,v,w_1,\cdots,w_{2t}}$ to $b^{\ell}_{w_j,z}$ for each $1 \le j \le 2t$.
- If $T = |T_0 \cup \cdots \cup T_{L-1}|$, then L is selected so that $\delta^2 L \ge T^{1-\delta}$.

Svensson's Construction



Svensson's Construction



$\Rightarrow$ indeg$(\hat{G}_{\mathcal{U}}) \geq L$ (and can be as large as $\Omega(N)$ in general.) Need to reduce indegree!

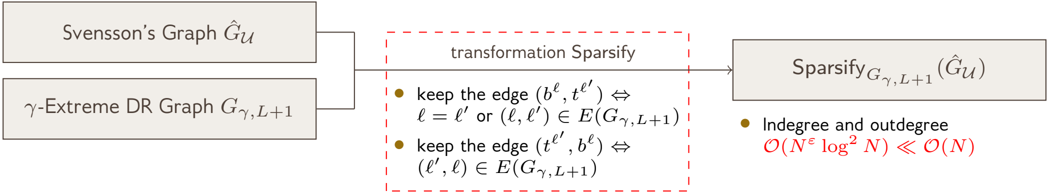
We are now at...

Technical Ingredients 2: γ-Extreme Depth Robust Graphs (Indegree Reduction)

- As discussed before, Svensson's construction has too large indegree ($\mathcal{O}(N)$) for the purposes of bounding $\mathrm{cc}(G)$. How to reduce indegree?

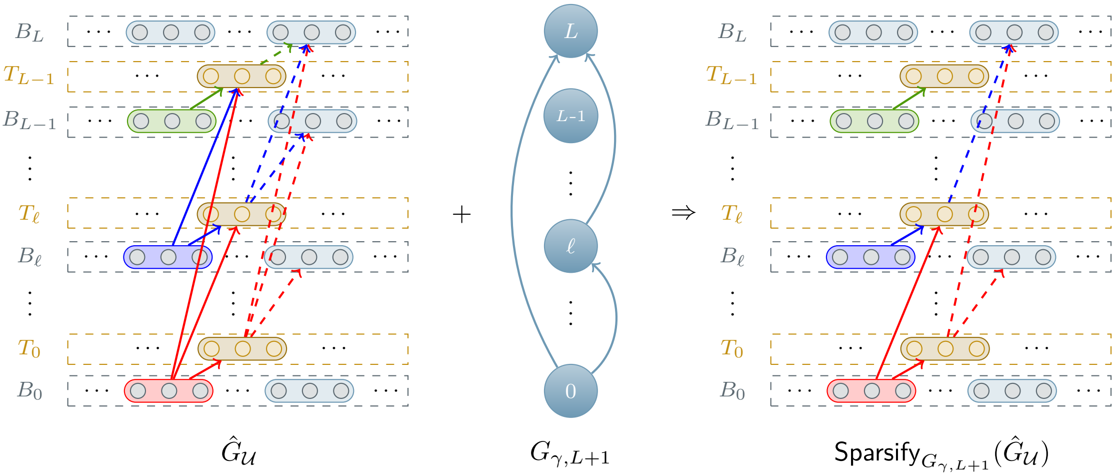
Definition

A DAG $G_{\gamma,N}$ on N nodes is said to be γ-***extreme depth-robust*** if it is (e,d)-depth robust for any $e, d > 0$ such that $e + d \leq (1-\gamma)N$.



- Alwen ***et al.*** [ABP18] showed that for any constant $\gamma > 0$, there exists a family $\{G_{\gamma,N}\}_{N=1}^{\infty}$ of γ-extreme depth-robust DAGs with maximum indegree and outdegree $\mathcal{O}(\log N)$.
- Then Sparsify$_{G_{\gamma,L+1}}(\hat{G}_{\mathcal{U}})$ will have degree at most $\mathcal{O}(\mathrm{indeg}(G_{\gamma,L+1}) \times \mathrm{outdeg}(G_{\gamma,L+1}) \times N/(L+1)) = \mathcal{O}(N^{\varepsilon} \log^2 N)$.

Technical Ingredients 2: γ-Extreme Depth Robust Graphs (Indegree Reduction)



Technical Ingredients 2: γ-Extreme Depth Robust Graphs (Indegree Reduction)

Theorem [Sve12]

For any integer $k \geq 2$ and constant $\varepsilon > 0$, it is Unique Games hard to distinguish between whether

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- Indegree Reduction with $\text{Sparsify}_{G_{\gamma, L+1}}(\hat{G}_{\mathcal{U}})$
- Analysis with Graph Coloring and Weighted Depth Robustness

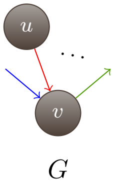
Theorem (3.3)

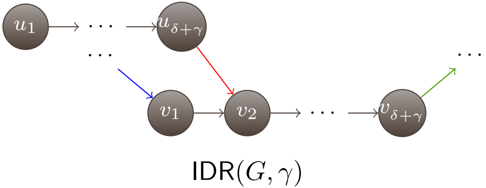
For any integer $k \geq 2$ and constant $\varepsilon > 0$, given a DAG G with N vertices and $\text{indeg}(G) = \mathcal{O}(N^{\varepsilon} \log^2 N)$, it is Unique Games hard to distinguish between the following cases:

- ***(Completeness): G is $\left(\left(\frac{1-\varepsilon}{k}\right) N, k\right)$-reducible.***
- ***(Soundness): G is $((1-\varepsilon)N, N^{1-\varepsilon})$-depth robust.***

Obtaining DAGs with Constant Indegree

- The second indegree reduction procedure $\mathsf{IDR}(G,\gamma)$ replaces each node $v \in V$ with a path $P_v = v_1, \cdots, v_{\delta+\gamma}$, where $\delta = \mathsf{indeg}(G)$.
- For each edge $(u,v) \in E$, we add the edge $(u_{\delta+\gamma}, v_j)$ whenever (u,v) is the j^{th} incoming edge of v.
- We observe that $\mathsf{indeg}(\mathsf{IDR}(G,\gamma)) = 2$.





Lemma ([ABP17])

- ***If** G **is** (e,d)**-reducible, then** $\mathsf{IDR}(G,\gamma)$ **is** $(e,(\delta+\gamma)d)$**-reducible.***
- ***If** G **is** (e,d)**-depth robust, then** $\mathsf{IDR}(G,\gamma)$ **is** $(e,\gamma d)$**-depth robust.***

Corollary (3.5)

For any integer $k \geq 2$ and constant $\varepsilon > 0$, given a DAG G with N vertices and $\text{indeg}(G) = 2$, it is Unique Games hard to decide whether G is (e_1, d_1)-reducible or (e_2, d_2)-depth robust for

- ***(Completeness):** $e_1 = \frac{1}{k} N^{\frac{1}{1+2\varepsilon}}$ **and** $d_1 = kN^{\frac{2\varepsilon}{1+2\varepsilon}}$.*
- ***(Soundness):** $e_2 = (1-\varepsilon) N^{\frac{1}{1+2\varepsilon}}$ **and** $d_2 = 0.9N^{\frac{1+\varepsilon}{1+2\varepsilon}}$.*

Proof Sketch. Suppose G' is a graph with M vertices. With setting $\gamma = M^{2\varepsilon} - \delta$,

$$G' \text{ with } M \text{ vertices} \quad \longrightarrow \quad G = \text{IDR}(G', \gamma) \text{ with } (\delta + \gamma)M = M^{1+2\varepsilon} = N \text{ vertices}$$

or equivalently, $M = N^{\frac{1}{1+2\varepsilon}}$. By the previous Lemma,

$d_1 = (\delta + \gamma)k$

- $G = \text{IDR}(G', \gamma)$ is (e_1, d_1)-reducible for $e_1 = \frac{M}{k} = \frac{N^{1/(1+2\varepsilon)}}{k}$ and $d_1 = kM^{2\varepsilon} = kN^{\frac{2\varepsilon}{1+2\varepsilon}}$.
- $G = \text{IDR}(G', \gamma)$ is (e_2, d_2)-depth robust for $e_2 = (1-\varepsilon)M = (1-\varepsilon)N^{1/(1+2\varepsilon)}$, while $d_2 = \gamma M^{1-\varepsilon} = (M^{2\varepsilon} - \delta)M^{1-\varepsilon}$. Since $\delta = \mathcal{O}(M^{\varepsilon} \log^2 M)$, for sufficiently large M, $d_2 = 0.9M^{1+\varepsilon} = 0.9N^{\frac{1+\varepsilon}{1+2\varepsilon}}$.

□

We are now at...

Technical Ingredients 3: Superconcentrators

Recall that we have the following upper and lower bounds for $\mathsf{cc}(G_{\mathcal{U}})$:

$$\mathsf{cc}(G_{\mathcal{U}}) \geq e_2 d_2, \text{ and}$$

$$\mathsf{cc}(G_{\mathcal{U}}) \leq \min_{g \geq d_1} \left(e_1 N + gN \times \mathsf{indeg}(G_{\mathcal{U}}) + \frac{N^2 d_1}{g} \right).$$

- Even after indegree reduction, still no gap between the pebbling complexity of the two cases.

$$e_1 N = \frac{1}{k} N^{\frac{1}{1+2\varepsilon}} N = \frac{1}{k} N^{\frac{2+2\varepsilon}{1+2\varepsilon}} \gg (1-\varepsilon) N^{\frac{2+\varepsilon}{1+2\varepsilon}} = e_2 d_2.$$

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→ Need to make it tighter!

Definition (Superconcentrator)

A ***superconcentrator*** is a graph that connects N input nodes to N output nodes so that any subset of k inputs and k outputs are connected by k vertex-disjoint paths for all $1 \leq k \leq N$. Moreover, the total number of edges in the graph should be $\mathcal{O}(N)$.

Technical Ingredients 3: Superconcentrator Overlay

Pippenger gives a superconcentrator with depth $\mathcal{O}(\log N)$.

Lemma ([Pip77])

There exists a superconcentrator G ***with at most*** $42N$ ***vertices, containing*** N ***input vertices and*** N ***output vertices, such that*** $\text{indeg}(G) \le 16$ ***and*** $\text{depth}(G) \le \log(42N)$.

Now we define the overlay of a superconcentrator on a graph G.

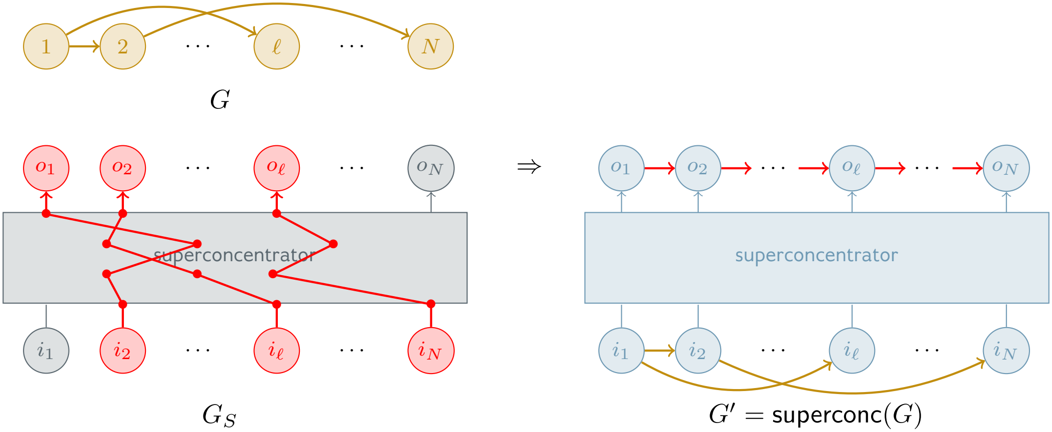
Definition (Superconcentrator Overlay)

Let $G = (V(G), E(G))$ be a fixed DAG with N vertices and $G_S = (V(G_S), E(G_S))$ be a (priori fixed) superconcentrator with N input vertices $\text{input}(G_S) = \{i_1, \cdots, i_N\} \subseteq V(G_S)$ and N output vertices $\text{output}(G_S) = \{o_1, \cdots, o_N\} \subseteq V(G_S)$. We call a graph $G' = (V(G_S), E(G_S) \cup E_I \cup E_O)$ a ***superconcentrator overlay*** where $E_I = \{(i_u, i_v) : (u, v) \in E(G)\}$ and $E_O = \{(o_i, o_{i+1}) : 1 \le i < N\}$ and denote as $G' = \text{superconc}(G)$.

- We will denote the interior nodes as $\text{interior}(G') = G' \setminus (\text{input}(G') \cup \text{output}(G'))$ where $\text{input}(G') = \text{input}(G_S)$ and $\text{output}(G') = \text{output}(G_S)$.

Technical Ingredients 3: Superconcentrator Overlay

Example.



Technical Ingredients 3: Superconcentrator Overlay

If G is (e,d)-depth robust, We have the following lower bound on the pebbling complexity from [BHK+18]:

$$\mathsf{cc}(\mathsf{superconc}(G)) \geq \min\left\{\frac{eN}{8}, \frac{dN}{8}\right\}.$$

The following lemma provides a ***significantly tighter*** upper bound on $\mathsf{cc}(\mathsf{superconc}(G))$ with an improved pebbling strategy.

Lemma (4.4)

Let G be an (e,d)-reducible graph with N vertices with $\mathsf{indeg}(G)=2$. Then

$$\mathsf{cc}(\mathsf{superconc}(G)) \leq \min_{g \geq d}\left\{2eN + 4gN + \frac{43dN^2}{g} + \frac{24N^2\log(42N)}{g} + 42N\log(42N) + N\right\}.$$

- Full description for the improved pebbling strategy: see the full paper! (Link)
- Now we can tune parameters appropriately to obtain our main result.

We are now at...

Main Theorem: Unique Games Hardness of $\mathsf{cc}(G)$

Theorem

Given a DAG G, it is Unique Games hard to approximate $\mathsf{cc}(G)$ within any constant factor.

Proof Sketch. Let $k \geq 2$ be an integer that we shall later fix. Similarly, $\varepsilon > 0$ be a constant that we shall later fix. Given a DAG G with N vertices, it is Unique Games hard to decide whether

- G is (e_1, d_1)-reducible for $e_1 = \frac{1}{k} N^{\frac{1}{1+2\varepsilon}}$, $d_1 = kN^{\frac{2\varepsilon}{1+2\varepsilon}}$, and
- G is (e_2, d_2)-depth robust for $e_2 = (1-\varepsilon) N^{\frac{1}{1+2\varepsilon}}$, $d_2 = 0.9 N^{\frac{1+\varepsilon}{1+2\varepsilon}}$.
- If G is (e_1, d_1)-reducible, then

$$\mathsf{cc}(\mathsf{superconc}(G)) \leq \min_{g \geq d_1} \left\{ 2e_1 N + 4gN + \frac{43 d_1 N^2}{g} + \frac{24 N^2 \log(42N)}{g} + 42N \log(42N) + N \right\}$$

$$\leq \frac{7}{k} N^{\frac{2+2\varepsilon}{1+2\varepsilon}} \quad \text{(for } g = e_1 \text{ and sufficiently large } N.)$$

- If G is (e_2, d_2)-depth robust, then $\mathsf{cc}(\mathsf{superconc}(G)) \geq \min\left\{\frac{e_2 N}{8}, \frac{d_2 N}{8}\right\} \geq \frac{1-\varepsilon}{8} N^{\frac{2+2\varepsilon}{1+2\varepsilon}}$.

Let $c \geq 1$ be any constant. Setting $\varepsilon = \frac{1}{2}$ and $k = 102c^2$, we have

$$\frac{7}{k} N^{\frac{2+2\varepsilon}{1+2\varepsilon}} = \frac{1}{16c^2} N^{\frac{2+2\varepsilon}{1+2\varepsilon}} \ll \frac{1}{16} N^{\frac{2+2\varepsilon}{1+2\varepsilon}} = \frac{1-\varepsilon}{8} N^{\frac{2+2\varepsilon}{1+2\varepsilon}}. \qquad \square$$

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- G is (e_2, d_2)-depth robust for $e_2 = (1-\varepsilon)N^{\frac{1}{1+2\varepsilon}}$, $d_2 = 0.9N^{\frac{1+\varepsilon}{1+2\varepsilon}}$.

(Corollary 3.5)

- If G is (e_1, d_1)-reducible, then

$$\mathsf{cc}(\mathsf{superconc}(G)) \leq \min_{g \geq d_1} \left\{ 2e_1 N + 4gN + \frac{43 d_1 N^2}{g} + \frac{24N^2 \log(42N)}{g} + 42N \log(42N) + N \right\}$$

(Lemma 4.4)

$$\leq \frac{7}{k} N^{\frac{2+2\varepsilon}{1+2\varepsilon}} \quad \text{(for } g = e_1 \text{ and sufficiently large } N.)$$

- If G is (e_2, d_2)-depth robust, then $\mathsf{cc}(\mathsf{superconc}(G)) \geq \min\left\{\frac{e_2 N}{8}, \frac{d_2 N}{8}\right\} \geq \frac{1-\varepsilon}{8} N^{\frac{2+2\varepsilon}{1+2\varepsilon}}$.

Let $c \geq 1$ be any constant. Setting $\varepsilon = \frac{1}{2}$ and $k = 102c^2$, we have

$$\frac{7}{k} N^{\frac{2+2\varepsilon}{1+2\varepsilon}} = \frac{1}{16c^2} N^{\frac{2+2\varepsilon}{1+2\varepsilon}} \ll \frac{1}{16} N^{\frac{2+2\varepsilon}{1+2\varepsilon}} = \frac{1-\varepsilon}{8} N^{\frac{2+2\varepsilon}{1+2\varepsilon}}.$$

□

We are now at...

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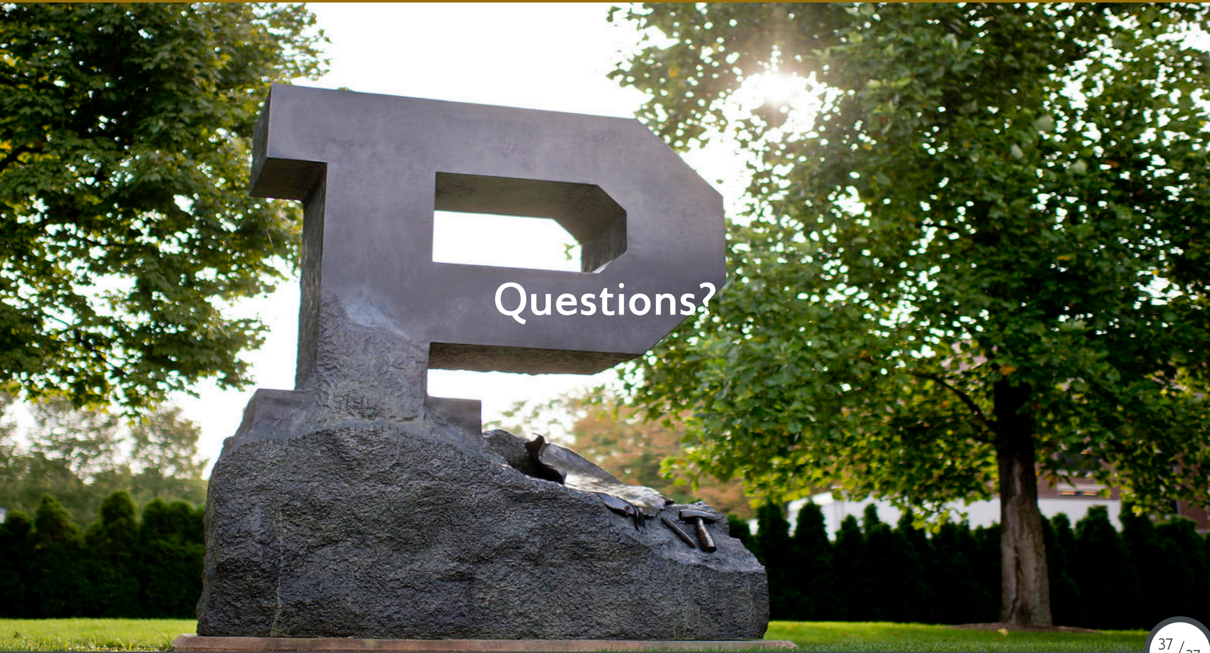
- Is there any efficient c-approximation algorithm for Red-Blue pebbling where $c = o(c_b/c_r)$?



Questions?